1D acoustic waves

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Newton's law

$$\rho\ddot{u} = -\frac{\partial P}{\partial z}$$

Where ρ is the density, u is the displacement, and P is the pressure. The gradient of the pressure, or in 1D the spatial derivative of the pressure is the force density, or force per volume element, because the force is the area times the negative of the difference of the pressure on the two opposing sides of a volume element. Therefore, the net force applied by the pressure on a volume element is the volume of the element times the pressure gradient. Since the mass is the volume times the density, the above equation is equivalent to the familiar form of Newton's law $m\ddot{u} = force$, where $\ddot{u}$ is the acceleration.

Hooke's law

$$P = -k\frac{\partial u}{\partial z}$$

P is the pressure, or the stress (in the acoustic case the stress tensor is diagonal). U is the displacement, and $\partial u/\partial z$ (or the divergence of the displacement in 3D) is the strain. The above equation simply states that the strain is proportional to the stress, and the proportion factor is the bulk modulus k. k is positive because positive pressure squeezes the material and creates a negative divergence. Since the strain is dimensionless, the dimensions of the bulk modulus are the same as the dimensions of pressure. $1/k$ is the compliance. Soft material has high compliance and low bulk modulus.

The wave equation

We can combine Newton's and Hooke's laws and get

$$\ddot{P} = -k\frac{\partial \ddot{u}}{\partial z} = k\frac{\partial}{\partial z}\frac{1}{\rho}\frac{\partial P}{\partial z}$$
$$\ddot{u} = -\frac{1}{\rho}\frac{\partial P}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}k\frac{\partial u}{\partial z}$$

In homogeneous medium, ρ and *k* are constants, so we have

$$\ddot{P} = \frac{k}{\rho}\frac{\partial^2 P}{\partial z^2}$$
$$\ddot{u} = \frac{k}{\rho}\frac{\partial^2 u}{\partial z^2}$$

Or, if we define the medium velocity $v = k/\rho$,

$$\ddot{P} = v^2 P_{zz}$$
$$\ddot{u} = v^2 u_{zz}$$

where u_{zz} and $\ddot{u}$ are the second derivatives of the displacement with space and time.

Waves

Waves are solutions to the above wave equations

$$P(t,z) = f(t \pm z/v)$$
$$u(t,z) = g(t \pm z/v)$$

If we differentiate by time and space we see that the above waves are solutions to the wave equations. This is true for both signs (plus or minus). If the z axis points down (into the earth) then the plus signs indicate up-going waves have and the minus signs indicate down-going waves.

How are the pressure and the displacement related?

The above waves P(t,z) and u (t.z) are independently solutions to the two wave equations. However, the two waves $f(t \pm z/v)$ and $g(t \pm z/v)$ are coupled via Newton's and Hooke's laws. Knowledge of *f()* implies knowledge of *g().*

Let's use Hooke's law,

$$P = -k\frac{\partial u}{\partial z}$$

Substituting $P(t,z) = f(t \pm z/v)$ and $u(t,z) = g(t \pm z/v)$ into Hooke's law gives

$$f(t \pm z/v) = -k\frac{\pm 1}{v}g'(t \pm z/v)$$

or using $v = \sqrt{k/\rho}$ we get

$$f(t \pm z/v) = \mp \rho v g'(t \pm z/v)$$

$\dot{u} = g'(t \pm z/v)$ (in time there is no issue of up and down sign) and $P = f$ so we have

$$P = \mp \rho v \dot{u}$$

ρv is called Impedance, $I = \rho v$.

$$P = \mp I\dot{u}$$

The $\mp$ means that the proportion between the pressure and the particle velocity is either plus the impedance for up-going waves or minus the impedance for down-going waves.

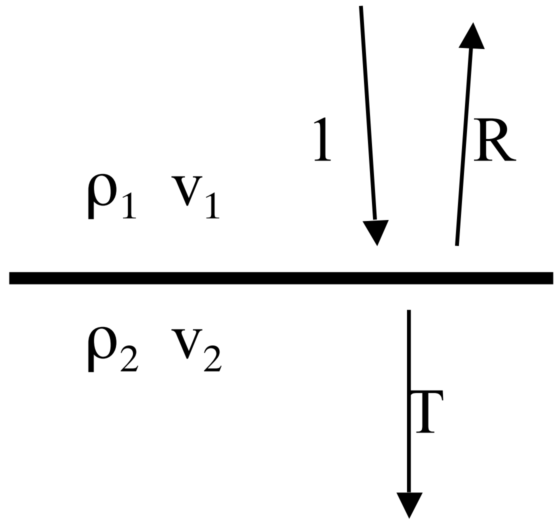
Up-down separation with accelerometer data

If the geophones or MEMS measure acceleration and velocity then the data are $\ddot{u}$ which is 90^0 phase shifted from $\dot{u}$, and $\pm 90^0$ phase shifted from P and we cannot simply add or subtract hydrophone and accelrometer data. But we can integrate the acceleration data $\ddot{u}$ to get velocity, $\dot{u}$ that can then be combined with pressure P, or to differentiate the pressure P to get $\dot{P}$ that can be combined with $\ddot{u}$, using

$$\dot{P} = \mp I\ddot{u}$$

Either way, the results (after post PZ-combination decon) will be the similar.

Reflection and transmission



The displacement wave is

$$u(t,z) = \begin{cases} g(t - z/v_1) + Rg(t + z/v_1) & z < 0 \\ Tg(t - z/v_2) & z > 0 \end{cases}$$

Using $P = \mp \rho v \dot{u}$ the pressure wave is

$$P(t,z) = \begin{cases} I_1\left(g'(t - z/v_1) - Rg'(t + z/v_1)\right) & z < 0 \\ I_2 T g'(t - z/v_2) & z > 0 \end{cases}$$

Continuity of displacement at z=0 means that

$$1 + R = T$$

Continuity of pressure at z=0 means that

$$I_1(1 - R) = I_2 T$$

So

$$R = \frac{I_1 - I_2}{I_1 + I_2}$$
$$T = \frac{2I_1}{I_1 + I_2}$$